Differential Geometry and its Applications 3 (1993) 323–329
North-Holland

Characteristic classes for G-structures*

D.V. Alekseevsky
Center "Sophus Lie", Krasnokazarmennaya 6, 111250 Moscow, Russia

P.W. Michor
Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria

Communicated by F. Tricerri
Received 10 July 1992

Alekseevesky D.V. and P.W. Michor, Characteristic classes for G-structures, Diff. Geom. Appl. 3 (1993) 323–329.

Abstract: Let $G \subset GL(V)$ be a linear Lie group with Lie algebra $\mathfrak{g}$ and let $A(\mathfrak{g})^G$ be the subalgebra of G-invariant elements of the associative supercommutative algebra $A(\mathfrak{g}) = S(\mathfrak{g}^*) \otimes \Lambda(V^*)$. To any G-structure $\pi : P \to M$ with a connection ω we associate a homomorphism $\mu_\omega : A(\mathfrak{g})^G \to \Omega(M)$. The differential forms $\mu_\omega(f)$ for $f \in A(\mathfrak{g})^G$ which are associated to the G-structure π can be used to construct Lagrangians. If ω has no torsion the differential forms $\mu_\omega(f)$ are closed and define characteristic classes of a G-structure. The induced homomorphism $\mu'_\omega : A(\mathfrak{g})^G \to H^*(M)$ does not depend on the choice of the torsionfree connection ω and it is the natural generalization of the Chern–Weil homomorphism.

Keywords: G-structures, characteristic classes.

MS classification: 53C10, 57R20.

1. G-structures

1.1. G-structures. By a G-structure on a smooth finite dimensional manifold M we mean a principal fiber bundle $\pi : P \to M$ together with a representation $\rho : G \to GL(V)$ of the structure group in a real vector space V of dimension $\dim M$ and a 1-form σ (called the *soldering form*) on M with values in the associated bundle $P[V,\rho] = P \times_G V$ which is fiber wise an isomorphism and identifies T_xM with $P[V]_x$ for each $x \in M$. Then σ corresponds uniquely to a G-equivariant 1-form $\theta \in \Omega^1_{\text{hor}}(P;V)^G$ which is strongly horizontal in the sense that its kernel is exactly the vertical bundle VP. The form θ is called the *displacement form* of the G-structure. A G-structure is called 1-*integrable* if it admits torsionfree connections, see 1.4 below.

We fix this setting $((P,p,M,G),(V,\rho),\theta)$ from now on.

1.2. Invariant forms. We consider a multilinear form $f \in \bigotimes^k V^* = L^k(V)$ which is

Correspondence to: P.W. Michor, Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria. *E-mail*: michor@awirap.bitnet.
*Supported by Project P 7724 PHY of 'Fonds zur Förderung der wissenschaftlichen Forschung'.

0926-2245/93/$06.00

invariant in the sense that $f\circ(\bigotimes^k\rho(g))=f$ for each $g\in G$. Let us denote by $L^k(V)^G$ the space of all these invariant forms. For each $f\in L^k(V)^G$ we have for any $X\in\mathfrak{g}$, the Lie algebra of G,

$$
\begin{aligned}
0&=\frac{d}{dt}\Big|_0 f(\rho(\exp(tX))v_1,\dots,\rho(\exp(tX))v_k)\\
&=\sum_{i=1}^k f(v_1,\dots,\rho'(X)v_i,\dots,v_k),
\end{aligned}
$$

where $\rho'=T_e\rho:\mathfrak{g}\to\mathfrak{gl}(V)$ is the differential of the representation ρ.

1.3. Products of differential forms. For $\varphi\in\Omega^p(P;\mathfrak{g})$ and $\Psi\in\Omega^q(P;V)$ let us define the form $\rho'_\wedge(\varphi)\Psi\in\Omega^{p+q}(P;V)$ by

$$
\begin{aligned}
&(\rho'_\wedge(\varphi)\Psi)(X_1,\dots,X_{p+q})\\
&\quad=\frac{1}{p!\,q!}\sum_\sigma \operatorname{sign}(\sigma)\rho'(\varphi(X_{\sigma1},\dots,X_{\sigma p}))\Psi(X_{\sigma(p+1)},\dots,X_{\sigma(p+q)}).
\end{aligned}
$$

Then $\rho'_\wedge(\varphi):\Omega^*(P;V)\to\Omega^{*+p}(P;V)$ is a graded $\Omega(P)$-module homomorphism of degree p. Recall also that $\Omega(P;\mathfrak{g})$ is a graded Lie algebra with the bracket

$$
\begin{aligned}
&[\varphi,\psi]_\wedge(X_1,\dots,X_{p+q})\\
&\quad=\frac{1}{p!\,q!}\sum_\sigma \operatorname{sign}\sigma\,[\varphi(X_{\sigma1},\dots,X_{\sigma p}),\psi(X_{\sigma(p+1)},\dots,X_{\sigma(p+q)})]_{\mathfrak{g}}.
\end{aligned}
$$

One may easily check that for the graded commutator in $\operatorname{End}(\Omega(P;V))$ we have

$$
\rho'_\wedge([\varphi,\psi]_\wedge)=[\rho'_\wedge(\varphi),\rho'_\wedge(\psi)]=\rho'_\wedge(\varphi)\circ\rho'_\wedge(\psi)-(-1)^{pq}\rho'_\wedge(\psi)\circ\rho'_\wedge(\varphi)
$$

so that $\rho'_\wedge:\Omega^*(P;\mathfrak{g})\to\operatorname{End}^*(\Omega(P;V))$ is a homomorphism of graded Lie algebras.

Let $\bigotimes V$ be the tensoralgebra generated by V. For $\Phi,\Psi\in\Omega(P;\bigotimes V)$ we will use the associative bigraded product

$$
\begin{aligned}
&(\Phi\otimes_\wedge\Psi)(X_1,\dots,X_{p+q})\\
&\quad=\frac{1}{p!\,q!}\sum_\sigma \operatorname{sign}(\sigma)\Phi(X_{\sigma1},\dots,X_{\sigma p})\otimes\Psi(X_{\sigma(p+1)},\dots,X_{\sigma(p+q)})
\end{aligned}
$$

1.4. The covariant exterior derivative. Let $\omega\in\Omega^1(P;\mathfrak{g})^G$ be a principal connection on the principal bundle (P,p,M,G). Let $\chi:TP\to HP$ denote the corresponding projection onto the horizontal bundle $HP:=\ker\omega$. The covariant exterior derivative $d_\omega:\Omega^k(P;V)\to\Omega^{k+1}_{\text{hor}}(P;V)$ is then given as usual by $d_\omega\Psi=\chi^*d\Psi=(d\Psi)\circ\Lambda^{k+1}(\chi)$.

Lemma. *For $\Psi\in\Omega_{\text{hor}}(P;V)^G$ the covariant exterior derivative is given by $d_\omega\Psi=d\Psi+\rho'_\wedge(\omega)\Psi$.*

Proof. If we insert one vertical vector field, say the fundamental vector field ζ_X for $X \in \mathfrak{g}$, into $d_\omega\Psi$, we get 0 by definition. For the right hand side we use $i_{\zeta_X}\Psi = 0$ and

$$\begin{aligned}\mathcal{L}_{\zeta_X}\Psi &= \frac{\partial}{\partial t}\Big|_0 (\mathrm{Fl}_t^{\zeta_X})^*\Psi = \frac{\partial}{\partial t}\Big|_0 \Psi \circ \Lambda^p(r^{\exp tX}) = \frac{\partial}{\partial t}\Big|_0 \rho(\exp(-tX))\Psi \\ &= -\rho'(X)\Psi\end{aligned}$$

to get

$$\begin{aligned}i_{\zeta_X}(d\Psi + \rho'_\wedge(\omega)\Psi) &= i_{\zeta_X}d\Psi + di_{\zeta_X}\Psi + \rho'_\wedge(i_{\zeta_X}\omega)\Psi - \rho'_\wedge(\omega)i_{\zeta_X}\Psi \\ &= \mathcal{L}_{\zeta_X}\Psi + \rho'_\wedge(X)\Psi = 0.\end{aligned}$$

Let now all vector fields ξ_i be horizontal, then we get

$$\begin{aligned}&(d_\omega\Psi)(\xi_0, \dots, \xi_k) = (\chi^* d\Psi)(\xi_0, \dots, \xi_k) = d\Psi(\xi_0, \dots, \xi_k), \\ &(d\Psi + \rho'_\wedge(\omega)\Psi)(\xi_0, \dots, \xi_k) = d\Psi(\xi_0, \dots, \xi_k). \quad \square\end{aligned}$$

1.5. Definition. If $\theta \in \Omega^1_{\mathrm{hor}}(P; V)^G$ is the displacement form of a G-structure then the *torsion* of the connection ω with respect to the G-structure is $\tau := d_\omega\theta = d\theta + \rho'_\wedge(\omega)\theta$.

Recall that a G-structure is called 1-integrable if it admits a connection without torsion. This notion has also been investigated in [9] where it was called prolongable.

1.6. Chern–Weil forms. For differential forms $\psi_i \in \Omega^{p_i}(P; V)$ and $f \in L^k(V) = (\bigotimes^k V)^*$ we can construct the following differential forms:

$$\begin{aligned}&\psi_1 \otimes_\wedge \dots \otimes_\wedge \psi_k \in \Omega^{p_1+\dots+p_k}(P; V \otimes \dots \otimes V), \\ &f^{\psi_1,\dots,\psi_k} := f \circ (\psi_1 \otimes_\wedge \dots \otimes_\wedge \psi_k) \in \Omega^{p_1+\dots+p_k}(P).\end{aligned}$$

The exterior derivative of the latter one is clearly given by

$$\begin{aligned}d(f \circ (\psi_1 \otimes_\wedge \dots \otimes_\wedge \psi_k)) &= f \circ d(\psi_1 \otimes_\wedge \dots \otimes_\wedge \psi_k) \\ &= f \circ \left(\textstyle\sum_{i=1}^k (-1)^{p_1+\dots+p_{i-1}} \psi_1 \otimes_\wedge \dots \otimes_\wedge d\psi_i \otimes_\wedge \dots \otimes_\wedge \psi_k\right).\end{aligned}$$

We also set $f^\psi := f^{\psi,\dots,\psi} = \operatorname{alt} f(\psi, \dots, \psi)$ for $\psi \in \Omega^p(P; V)$. Note that the form $f^{\psi_1,\dots,\psi_k}$ is G-invariant and horizontal if all $\psi_i \in \Omega^{p_i}_{\mathrm{hor}}(P; V)^G$ and $f \in L^k(V)^G$. It is then the pullback of a form on M.

1.7. Lemma. *Let $0 \neq \psi \in \Omega^p(P; V)$ and $f \in L^k(V)$. Then we have:*

$$f^\psi \neq 0 \iff \begin{cases} \operatorname{alt} f \neq 0, & \textit{if } p \textit{ is odd}, \\ \operatorname{sym} f \neq 0, & \textit{if } p \textit{ is even}, \end{cases}$$

where alt *and* sym *are the natural projections onto $\Lambda(V^*)$ and $S(V^*)$, respectively.* $\square$

1.8. Lemma. *If $f \in L^k(V)^G$ is invariant then we have*

$$f \circ \Big(\sum_{i=1}^k (-1)^{p_1+\dots+p_{i-1}} \psi_1 \otimes_\wedge \dots \otimes_\wedge \rho'_\wedge(\omega)\psi_i \otimes_\wedge \dots \otimes_\wedge \psi_k \Big) = 0.$$

Proof. This follows from the infinitesimal condition of invariance for f given in 1.2 by applying the alternator. □

2. Obstructions to 1-integrability of G-structures

2.1. Proposition. *Let $\pi : P \to M$ be a G-structure and let $f \in L^k(V)^G$ be an invariant tensor. For arbitrary G-equivariant horizontal V-valued forms $\psi_i \in \Omega^{p_i}_{\text{hor}}(P;V)^G$ we consider the $(p_1 + \dots + p_k)$-form $f^{\psi_1,\dots,\psi_k}$ on M as above. If there is a connection ω for the G-structure π such that $d_\omega\psi_i = 0$ for all i, then the form $f^{\psi_1,\dots,\psi_k}$ is closed.*

Proof. We use $d_\omega\psi_i = d\psi_i + \rho'_\wedge(\omega)\psi_i$ from Lemma 1.4, and Lemma 1.8, to obtain

$$df^{\psi_1,\dots,\psi_k} = f \circ \Big(\sum_{i=1}^k (-1)^{p_1+\dots+p_{i-1}} \psi_1 \otimes_\wedge \dots \otimes_\wedge d_\omega\psi_i \otimes_\wedge \dots \otimes_\wedge \psi_k \Big) = 0. \quad \square$$

2.2. Corollary. 1. *For a G-structure $\pi : P \to M$ with displacement form θ we have a natural homomorphism of associative algebras*

$$\nu : \Lambda(V^*)^G \to \Omega(M),$$
$$f \mapsto f^\theta = f(\theta, \dots, \theta).$$

2. *If the G-structure is 1-integrable then the image of ν consists of closed forms and we get an induced homomorphism*

$$\nu^* : \Lambda(V^*)^G \to H^*(M).$$

If M and G are compact then ν^* is injective.

Proof. If the G-structure is 1-integrable then there is a connection ω with vanishing torsion $\tau = d_\omega\theta = 0$. Then the result follows from Proposition 2.1.

If G is compact, any torsionfree connection ω for $\pi : P \to M$ is the Levi-Civita connection for some Riemannian metric. Any form f^θ, which is parallel with respect to ω, is harmonic and can thus not be exact for compact M. So ν^* is injective. □

Problem. Is the homomorphism ν^* injective for compact M but noncompact G?

2.3. Remark. Given a principal connection ω on P there is the induced covariant exterior derivative $\nabla : \Omega^p(M; P[V]) \to \Omega^{p+1}(M; P[V])$ on the associated vector bundle $P[V]$. The soldering form (see 1.1) $\sigma : TM \to P[V]$ is an isomorphism of vector bundles and we may consider the pull back covariand derivative $\sigma^*\nabla$ on TM. Next we consider the 'combined' covariant derivative $D^{\sigma^*\nabla,\nabla}$ on the vector bundle $L(TM, P[V])$ given by $D^{\sigma^*\nabla,\nabla}_X A = \nabla_X \circ A - A \circ (\sigma^*\nabla)_X$. Obviously we have $D^{\sigma^*\nabla,\nabla}\sigma = 0$. Consequently for any $f \in L^k(V)^G$ we have that $f^\theta \in \Omega^k(M)$ is parallel for the connection induced on $\Lambda^k T^*M$ from $\sigma^*\nabla$ on TM.

3. The generalized Chern–Weil homomorphism for G-structures

3.1. The Chern–Weil homomorphism. Let ω be a connection for a G-structure $\pi : P \to M$ with curvature form $\Omega \in \Omega^2_{\mathrm{hor}}(P, \mathfrak{g})$. Then the Bianchi identity $d_\omega \Omega = 0$ holds. If we apply Proposition 2.1 to $\psi_i = \Omega$ we obtain a homomorphism

$$\gamma : S(\mathfrak{g}^*)^G \to \Omega(M),$$

given by $\gamma(f) = f^\Omega$. Since the image of γ consists of closed forms we have an induced homomorphism

$$\gamma' : S(\mathfrak{g}^*)^G \to H^*(M).$$

This is the well known Chern–Weil homomorphism.

3.2. The algebra $A(\mathfrak{g}, V)$. In oder to generalize the Chern–Weil homomorphism we associate to a Lie algebra $\mathfrak{g}$ and a vector space V the associative graded commutative algebra

$$A(\mathfrak{g}, V) := S(\mathfrak{g}^*) \otimes \Lambda(V^*),$$

where the generators of the symmetric algebra $S(\mathfrak{g}^*)$ have degree 2. We may also consider $A(\mathfrak{g}, V)$ as a graded Lie algebra with the bracket

$$[a \otimes \varphi, b \otimes \psi] := \{a, b\} \otimes \varphi \wedge \psi, \quad a, b \in S(\mathfrak{g}^*), \varphi, \psi \in \Lambda(V^*),$$

where $\{a, b\}$ is the usual Poisson-Lie bracket in $S(\mathfrak{g}^*)$.

Let now $\mathfrak{g}$ be the Lie algebra of the Lie group G and let $\rho : G \to GL(V)$ be a representation. Then G acts naturally on $A(\mathfrak{g}, V)$, and we denote $A(\mathfrak{g}, V)^G$ the subalgebra of G-invariant elements in $A(\mathfrak{g}, V)$.

3.3. Remark. The associative algebra $A(\mathfrak{g}, V)^G$ contains the subalgebra $S(\mathfrak{g}^*)^G \otimes \Lambda(V^*)^G$, in general as a proper subalgebra. Actually, let $G \subset GL(V)$ be the isotropy group of an irreducible Riemannian symmetric space M. Then the curvature tensor of M defines an element of $(\mathfrak{g}^* \otimes \Lambda^2 V^*)^G \subset A(\mathfrak{g}, V)^G$ that does not belong to $(\mathfrak{g}^*)^G \otimes (\Lambda^2 V^*)^G = 0$.

3.4. The generalized Chern–Weil homomorphism. Now we are in a position to combine the Constructions 2.2 and 3.1.

Theorem. *Let $\pi : P \to M$ be a G-structure on M with displacement form θ. Any connection ω in π defines a homomorphism of associative algebras*

$$\mu : A(\mathfrak{g}, V)^G \to \Omega(M),$$
$$(S^p(\mathfrak{g}^*) \otimes \Lambda^q V^*)^G \ni f \mapsto f^{\Omega,\theta} = f(\underbrace{\Omega, \dots, \Omega}_{p}, \underbrace{\theta, \dots, \theta}_{q}).$$

If the connection ω has no torsion then the image of μ consists of closed forms and μ induces a homomorphism

$$\mu' : A(\mathfrak{g}, V)^G \to H^*(M),$$

which is independent of the choice of the torsionfree connection.

In other words, any G-invariant tensor $f \in S^p(\mathfrak{g}^*) \otimes \Lambda^q(V^*)$ defines a cohomology class $[f^{\Omega,\theta}] \in H^{2p+q}(M)$ which is an invariant of the 1-integrable G-structure. We call it a *characteristic class* of the 1-integrable G-structure π.

Proof. It just remains to show that the cohomology class $[f^{\Omega,\theta}]$ does not depend on the choice of the torsionfree connection for the G-structure $\pi : P \to M$.

So let ω_0, ω_1 be two torsionfree connections for the G-structure, let $\varphi = \omega_1 - \omega_0$, and denote by $\Omega_t = d_{\omega_t}\Omega_t$ the curvature form of the torsionfree connection $\omega_t = \omega_0 + t\varphi = (1-t)\omega_0 + t\omega_1$. We claim that for $f \in (S^p(\mathfrak{g}^*) \otimes \Lambda^q(V^*))^G$ we have

$$f^{\Omega_1,\theta} - f^{\Omega_0,\theta} = d(Tf), \quad \text{where} \tag{1}$$

$$Tf = p\int_0^1 f(\varphi, \Omega_t, \dots, \Omega_t, \theta, \dots, \theta)\, dt$$

is the transgression form of f on P. The assertion is immediate from (1). To prove it we compute $\partial_t f^{\Omega_t,\theta}$ using the identities $\partial_t\Omega_t = d_{\omega_t}\varphi$ (see [7,II,p.296]), $d_{\omega_t}\Omega_t = 0$, and $d_{\omega_t}\theta = 0$:

$$\begin{aligned}
\partial_t f^{\Omega_t,\theta} &= p\, f(\partial_t\Omega_t, \Omega_t, \dots, \Omega_t, \theta, \dots, \theta)\\
&= p\, f(d_{\omega_t}\varphi, \Omega_t, \dots, \Omega_t, \theta, \dots, \theta)\\
&= p\, d_{\omega_t} f(\varphi, \Omega_t, \dots, \Omega_t, \theta, \dots, \theta)\\
&= p\, d\, f(\varphi, \Omega_t, \dots, \Omega_t, \theta, \dots, \theta). \quad \square
\end{aligned}$$

3.5. Remarks about secondary characteristic classes. If the characteristic forms $f^{\Omega_1,\theta}$ and $f^{\Omega_0,\theta}$ associated with two torsionfree connections ω_1 and ω_0 vanish we obtain a secondary characteristic class $[Tf]$. It is a natural generalization of the classical Chern–Simons characteristic class, see [4, 5, 8].

Problem: study conditions when the secondary characteristic class $[Tf]$ does not depend on the choice of the torsionfree connections ω_1 and ω_0.

3.6. Examples of characteristic classes. Assume that a linear group $G \subset GL(V)$ preserves some pseudo Euclidean metric in $V = \mathbb{R}^n$. Then we may identify the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ with a subspace $\mathfrak{g} \subset \Lambda^2 V$. Suppose that there exists a G-invariant supplement $\mathfrak{d}$ to $\mathfrak{g}$ in $\Lambda^2 V$. Then the G-equivariant projection $\Lambda^2 V \to \mathfrak{g}$ along $\mathfrak{d}$ determines a G-invariant element $q \in \mathfrak{g} \otimes \Lambda^2 V^* \cong \mathfrak{g}^* \otimes \Lambda^2 V^*$. The element q defines a 4-form $q^{\Omega,\theta}$ on the base of any G-structure $\pi : P \to M$ with a connection ω and curvature Ω. It may be written as

$$q^{\Omega,\theta} = q(\Omega, \theta, \theta) = q^a_{bcd} R^b_{aef} \theta^c \wedge \theta^d \wedge \theta^e \wedge \theta^f,$$

where (q^a_{bcd}) is the coordinate expression of q in the standard basis (e_a) of $V = \mathbb{R}^n$, $\theta = e_a \otimes \theta^a$, and $\Omega = R^a_{bef}\theta^e \wedge \theta^f$.

If ω is torsionfree the 4-form $q^{\Omega,\theta}$ is closed and it defines a cohomology class $[q^{\Omega,\theta}] \in H^4(M)$ independently of the choice of ω.

3.7. Remarks about the classification of characteristic classes. The classification of charactericic classes for G-structures with a given Lie group G reduces to the construction of generators of the associative algebra $A(\mathfrak{g}, V)^G = (S(\mathfrak{g}^*) \otimes \Lambda(V^*))^G$. We may also use the bracket to multiply characteristic classes. It suffices to solve this problem for those Lie groups G which appear as holonomy groups of torsionfree connection. Only for such groups G there exist 1-integrable non-flat G-structures. Under the additional hypothesis of irreducibility, all such groups were classified by [2], up to some gaps which were filled by [3, 1].

References

[1] D.V. Alekseevsky and M.M. Graev, Twistors and G-structures, Preprint, University of Roma.

[2] M. Berger, Sur les groupes d'holonomie des variétés à connexion affine et des variétés riemanniénnes, *Bull. Soc. Math. France 83* **155** (1955) 279–330.

[3] R. Bryant, Holonomy groups, Lecture at the conference: Global Differential Geometry and Global Analysis, Berlin 1990.

[4] S.S. Chern and J. Simons, Some cohomology classes in principal fiber bundles and their applications to Riemannian geometry, *Proc. Nat. Acad. Sci. (USA)* **68** (1971) 791–794.

[5] S.S. Chern and J. Simons, Charateristic forms and geometrical invariants, *Annals of Math.* **99** (1974) 48–69.

[6] Dupont and L. Johan, *Curvature and characteristic classes*, Lecture Notes in Mathematics **640** (Springer-Verlag, Berlin, 1978).

[7] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, Vol. II* (J. Wiley-Interscience, 1969).

[8] S. Kobayashi and T. Ochiai, G-structures of order two and transgression operators, *J. Diff. Geom.* **6** (1971) 213–230.

[9] I. Kolář and I. Vadovičová, On the structure functions of a G-structure, *Math. Slovaca* **35** (1985) 277–282.